3.3 Cramer's rule & the geometry of the determinant

Key idea: After discussing a useful tool, we turn to understanding what a determinant tells us geometrically. We will see that the determinant gives us the factor by which a 1x1 square's area changes under the transformation determined by the matrix in question.

Before discussing the geometric information inherit in the determinant we briefly overview a useful theoretic tool: Cramer's rule.

Fix an $n \times n$ matrix A and for any $\vec{b}$ in $\mathbb{R}^n$ define a new matrix

$$A_i(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{b} & \cdots & \vec{a}_n \end{bmatrix}$$

↳ column i

by replacing the ith column of A with $\vec{b}$. Then the following is true:

Cramer's rule: If A is an invertible $n \times n$ matrix and $\vec{b}$ in $\mathbb{R}^n$, then the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ has entries of the form

$$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)}, \quad i = 1, \dots, n.$$

We will see an example of this below but for high dimension, Cramer's rule is woefully inefficient at calculating solutions. It is useful theoretically though, as from Cramer's rule we can derive a formula (not algorithm) for the inverse of A. (See pg 181)

Ex 1 Use Cramer's rule to solve
$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$
so $A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$.

So $A_1(\vec{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}$, $A_2(\vec{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$ and by Cramer's rule:

$$x_1 = \frac{\det A_1(\vec{b})}{\det(A)} = \frac{24+16}{12-10} = \frac{40}{2} = 20.$$

$$x_2 = \frac{\det A_2(\vec{b})}{\det(A)} = \frac{24+30}{2} = \frac{54}{2} = 27$$

So $x_1 = 20$ and $x_2 = 27$ is the solution to the system.

"Now for the focus of this section:"

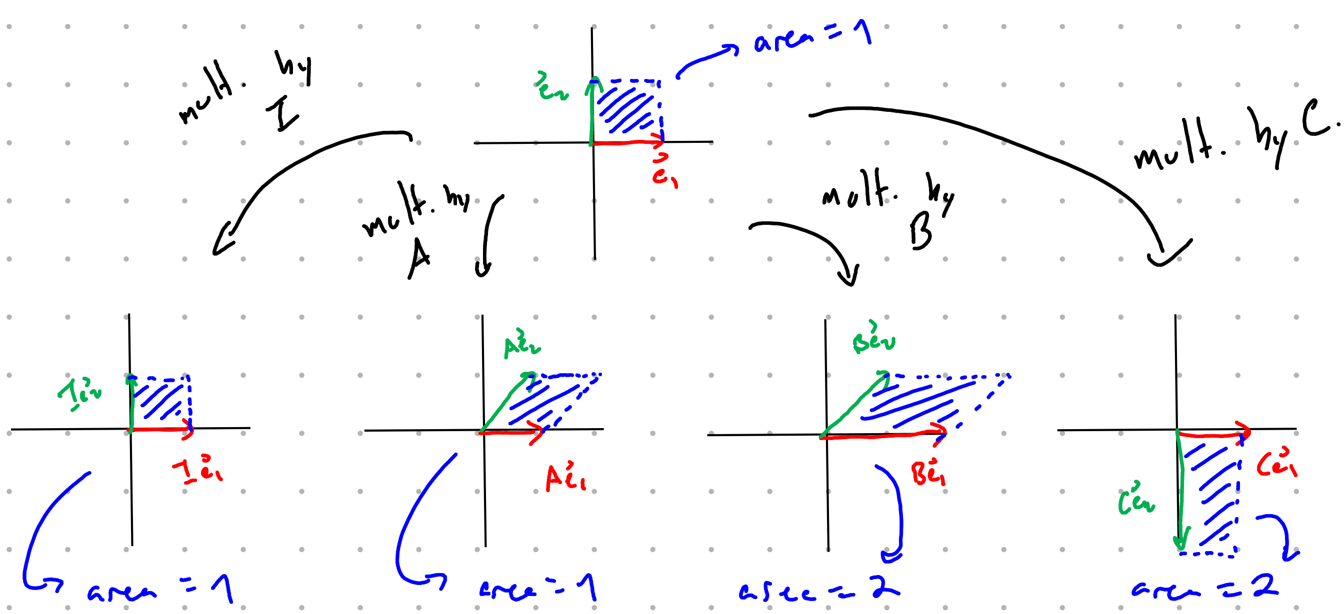
Geometry of the determinant.

We will see that $\det(A)$ informs us about how area changes under the linear transformation defined by A.

Consider the matrices

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

and their action on the square spanned by $\vec{e}_1, \vec{e}_2$ in $\mathbb{R}^2$:



$$\det(I) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \det(A) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad \det(B) = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 \qquad \det(C) = \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2$$

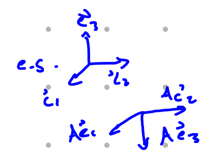
"So we see the determinant tells us the area of the parallelogram spanned by the columns of the matrix (in absolute value!). This is true in general."

Fact: If A is a 2×2 matrix, then $|\det(A)|$ is the area of the parallelogram spanned by the columns of A. If A is 3×3, $|\det(A)|$ is the area of the parallelopiped spanned by the columns of A.

Ex $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 2 \\ 3 & 1 & -2 \end{bmatrix}$. Note $A\vec{e}_1 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, $A\vec{e}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $A\vec{e}_3 = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$ and $\det(A) = -36$.

So the volume of the parallelopiped spanned by the columns of A is $|\det(A)| = -36$.

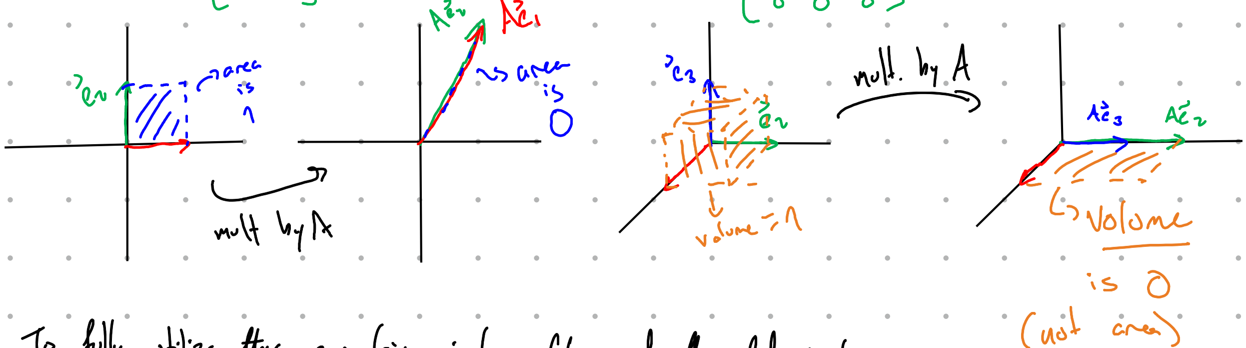
Consider "Volume under a linear transformation" on the course webpage
We see $\det(A) < 0$ corresponds to changing the orientation of $\vec{e}_1, \vec{e}_2, \vec{e}_3$



Fact: A is invertible if and only if $\det(A) \neq 0$.

Why? $\det(A) = 0$ means the area spanned by the two columns of A is 0
or the volume spanned by the three columns of A is 0.
This indicates A "squishes" 3D down to 1D or 2D for instance, so $T(\vec{x}) = A\vec{x}$ can't be onto, and thus A is not invertible.

Ex $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ has $\det(A) = 0$. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $\det(A) = 0$.



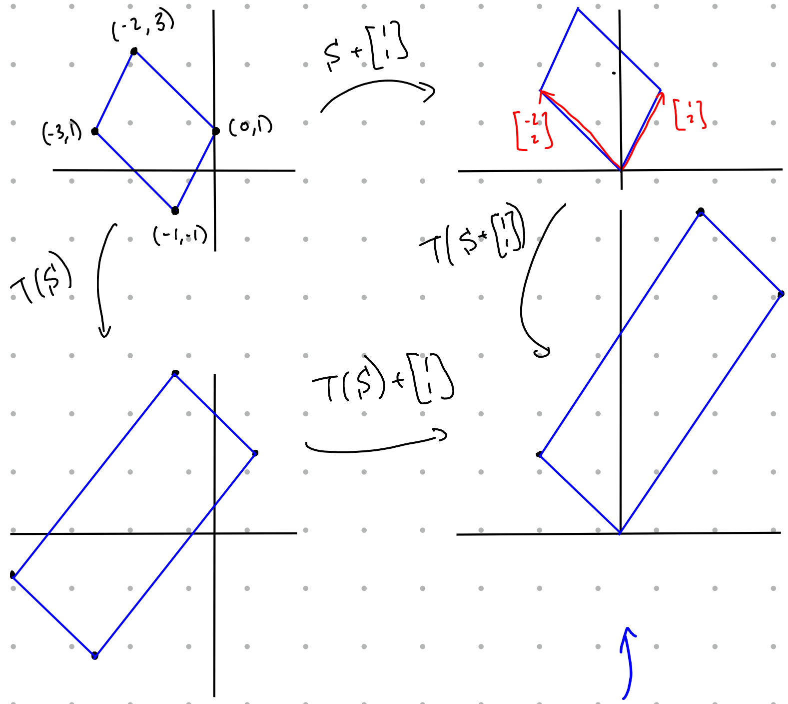
To fully utilize this geometric interpretation of the determinant we generalize this relationship between the area of the "unit square" and any parallelogram in the plane.

Fact: If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is determined by a 2×2 matrix A for every parallelogram in $\mathbb{R}^2$ S, we have

$$\text{"area of } T(S)\text{"} = |\det A| \cdot \text{"area of } S\text{"}$$

The same is true for $T: \mathbb{R}^3 \to \mathbb{R}^3$, A 3×3, and S a parallelopiped in $\mathbb{R}^3$.
(In fact, we can use this equation to understand 4, 5 and n-dimensional volume!)

Ex| Consider the parallelogram S with vertices $(-1,-1)$, $(0,1)$, $(-2,3)$ and $(-3,1)$. Compute the area of S and $T(S)$ if $T:\mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation $T(\vec{x}) = A\vec{x}$ with $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$



Notice if we translate S to a parallelogram with a vertex on the origin, we can compute it's area with the determinant:

$$\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} = 6$$

so area of S is 6.

By the above fact

$$\text{area of } T(S) = |\det A| \cdot 6 = \left|\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\right| \cdot 6 = 3 \cdot 6 = \boxed{18}$$

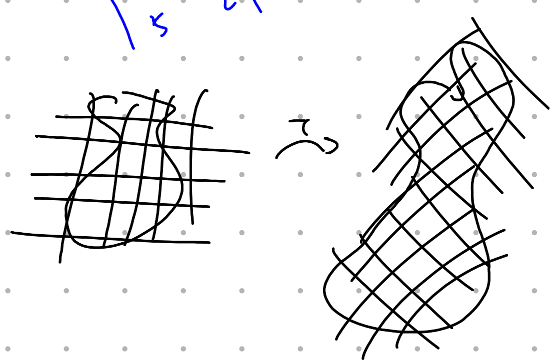
notice this is the same as area of $T(S) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

$T(S) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is spanned by $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ and

$$\begin{vmatrix} 4 & -2 \\ 5 & 2 \end{vmatrix} = 18.$$

"Why do we care about this?

- approximating non-parallelograms by many small parallelograms allow us to estimate any area and how it is changed under a transformation
- this is exactly how we compute integrals by change of variables the transformation in question is the change of variables and the determinant which measures change in area is the Jacobian"



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